Introduction to discrete-time systems

Characterisation and properties

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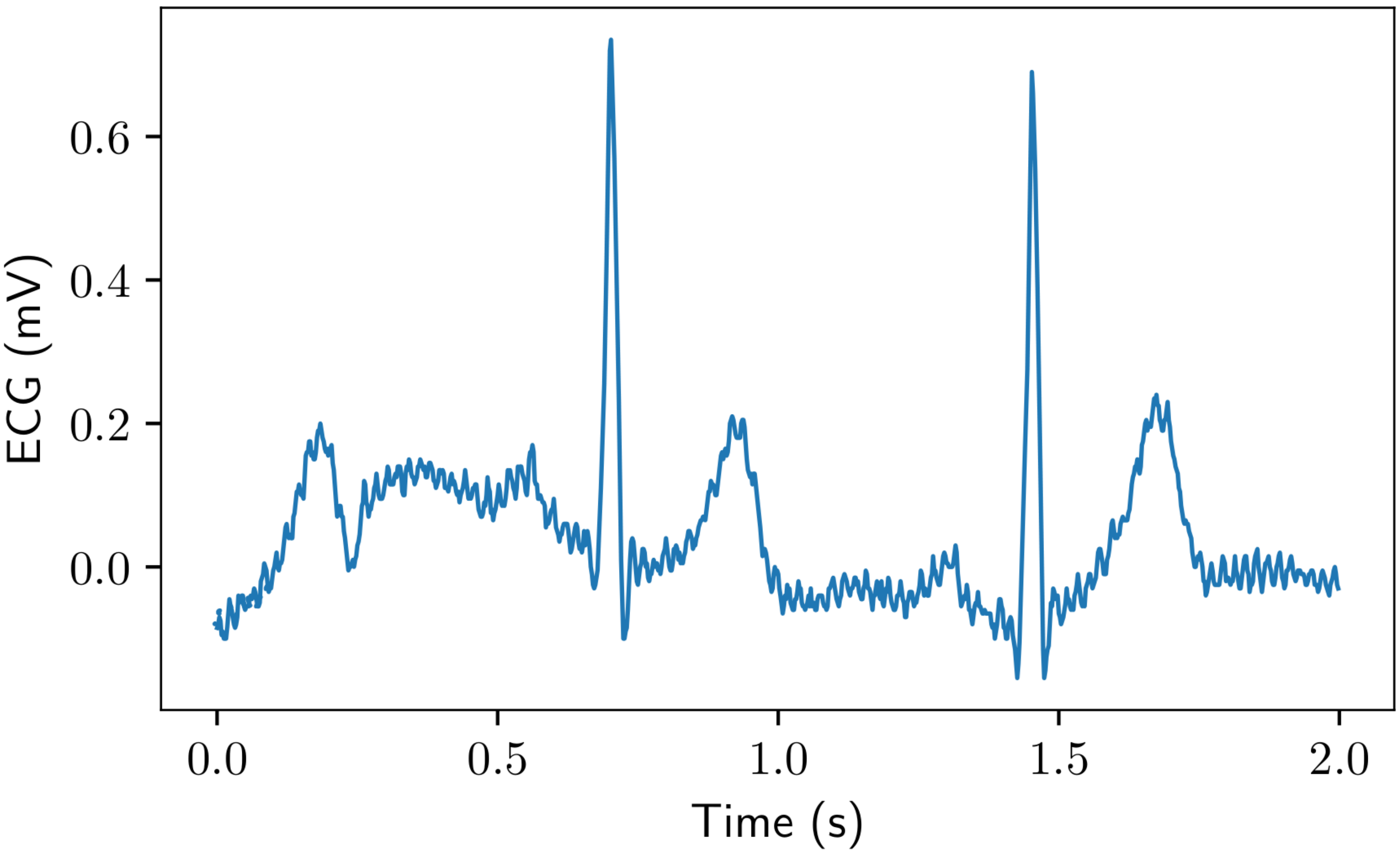
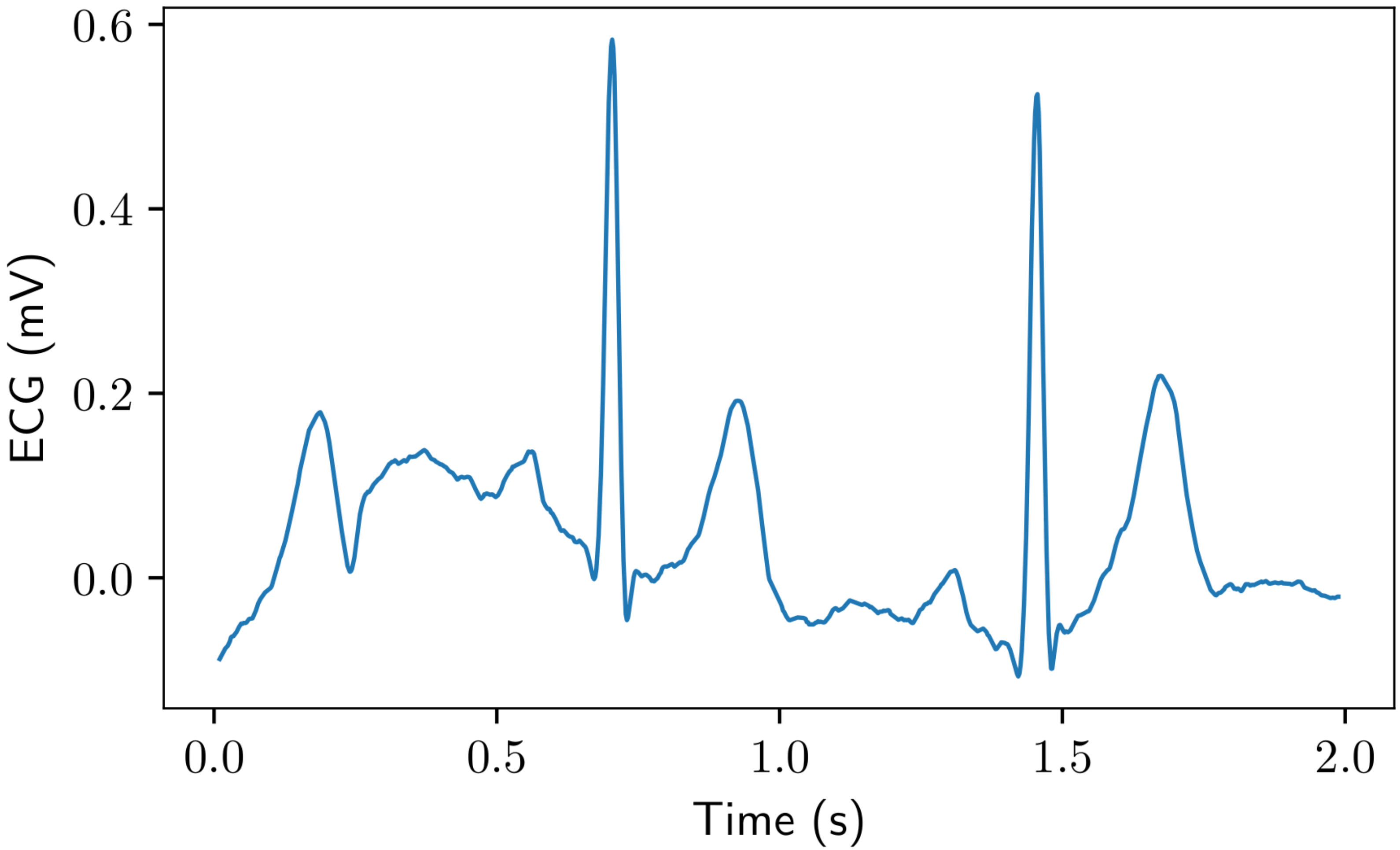


ECG (mV)
0.6
0.4
0.2
0.0
0.0
0.5
1.0
1.5
2.0
Time (s)



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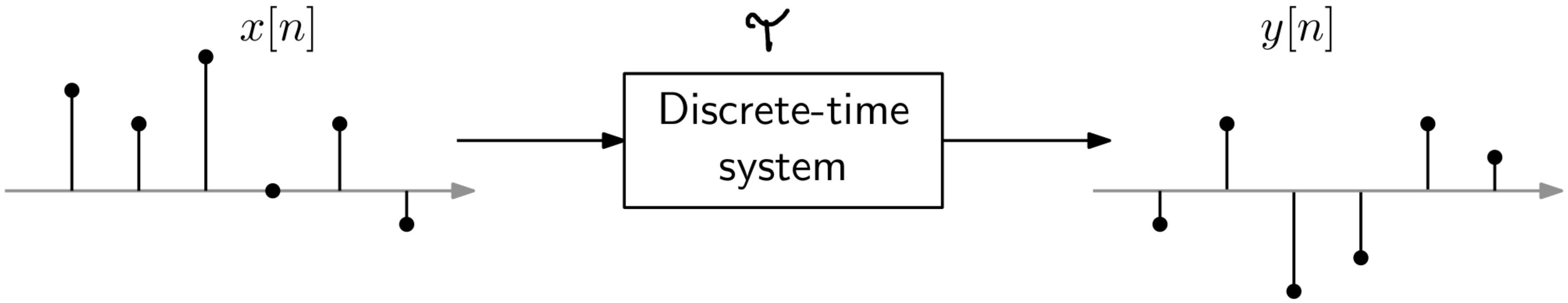
0.4
0.2
0.0
ECG (mV)
0.0
0.5
1.0
1.5
2.0
Time (s)

$x[n] \longrightarrow \boxed{\text{smooth}(\cdot)} \longrightarrow y[n]$

```python
def smooth(x):
    y = []
    window = 10
    for i in range(len(x) - window):
        y.append(1 / window * np.sum(x[i : i + window]))
    return y
```



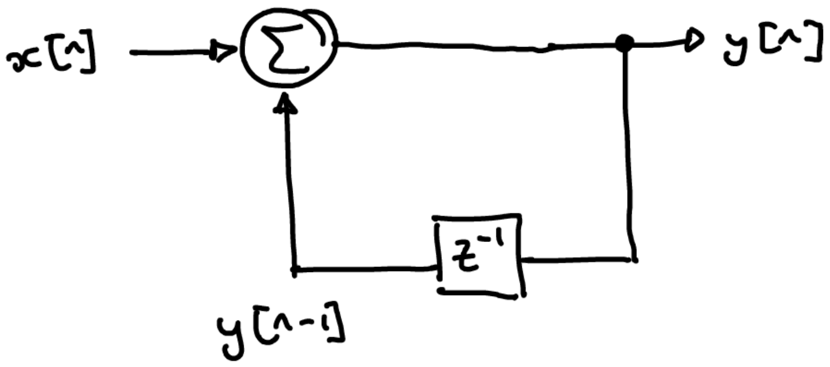
$$y[n] = \mathcal{T}\{x[n]\}$$

Accumulator:

Example: $y[n] = \sum_{k=-\infty}^{n} x[k]$

$= \sum_{k=-\infty}^{n-1} x[k] + x[n]$

$= y[n-1] + x[n]$



Delay: z^{-1} z^{-k}

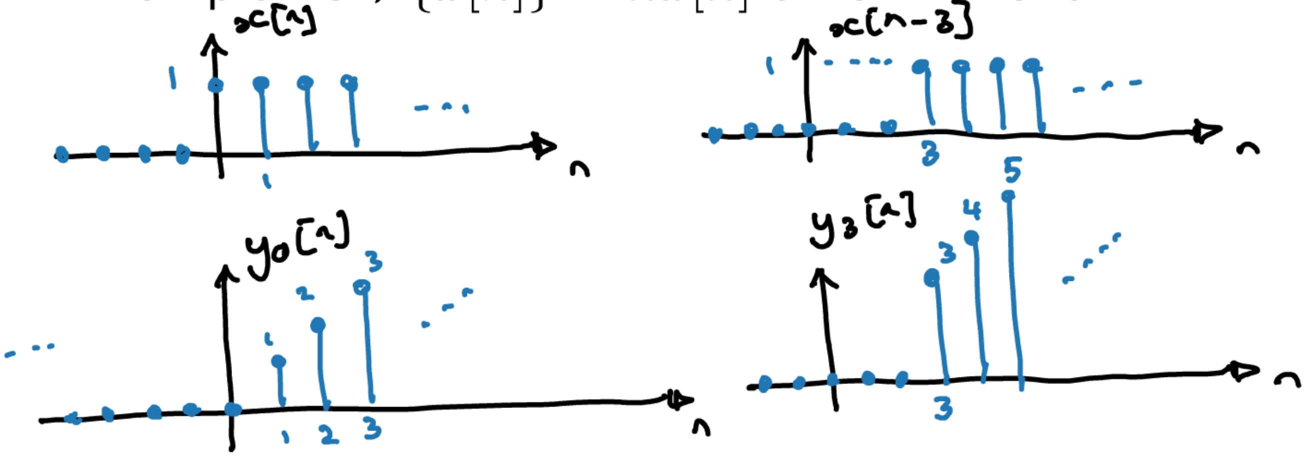
Advance: z^{+1} z^{k}

Time-invariant and time-variant systems

Time-invariant system:

$$\text{if} \qquad \mathcal{T}\{x[n]\} = y[n]$$
$$\text{then} \qquad \mathcal{T}\{x[n-k]\} = y[n-k]$$

Example: Is $\mathcal{T}\{x[n]\} = nx[n]$ time invariant?



$\neq y_0[n-3]$

$y_0[n] = \mathcal{T}\{x[n]\} = n\,x[n]$

$y_k[n] = \mathcal{T}\{x[n-k]\} = n \cdot x[n-k]$

But $y_0[n-k] = (n-k) \cdot x[n-k]$

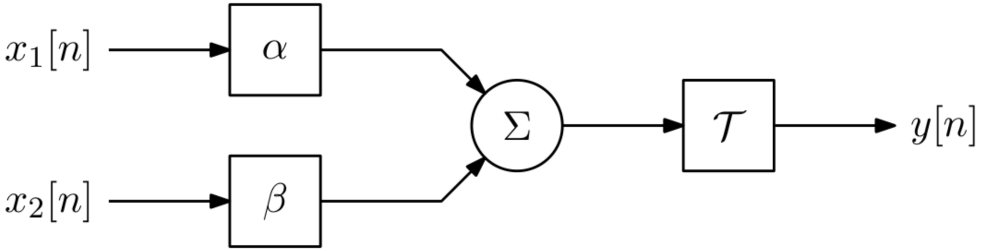
$\neq n \cdot x[n-k]$

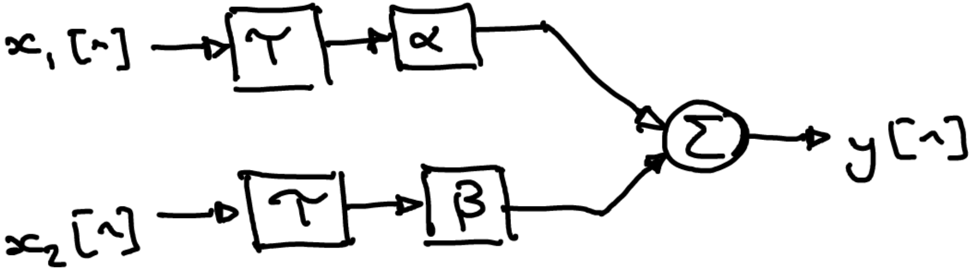
$\therefore$ time variant

Linear systems

Linear systems obey the principle of superposition:

$$\mathcal{T}\left\{\alpha x_1[n] + \beta x_2[n]\right\} = \alpha \mathcal{T}\left\{x_1[n]\right\} + \beta \mathcal{T}\left\{x_2[n]\right\}$$





In general:

$$\boxed{\mathcal{T}\left\{\sum_{i=1}^{N} \alpha_i x_i[n]\right\} = \sum_{i=1}^{N} \alpha_i \mathcal{T}\left\{x_i[n]\right\}}$$

Is $\mathcal{T}\{x[n]\} = nx[n]$ a linear system?

$$\alpha\, \mathcal{T}\{x_1[n]\} = \alpha n x_1[n]$$

$$\begin{aligned}
&\mathcal{T}\{\alpha x_1[n] + \beta x_2[n]\} \\
&= n \cdot (\alpha x_1[n] + \beta x_2[n]) \\
&= \alpha n x_1[n] + \beta n x_2[n] \\
&= \alpha \mathcal{T}\{x_1[n]\} + \beta \mathcal{T}\{x_2[n]\}
\end{aligned}$$

$\therefore$ Linear

Is $\mathcal{T}\{x[n]\} = x^2[n]$ a linear system?

$$\begin{aligned}
&\mathcal{T}\{\alpha x_1[n] + \beta x_2[n]\} \\
&= (\alpha x_1[n] + \beta x_2[n])^2 \\
&= \alpha^2 x_1^2[n] + 2\alpha\beta x_1[n] \cdot x_2[n] + \beta^2 x_2^2[n] \quad \ldots (1)
\end{aligned}$$

$$\begin{aligned}
&\alpha \mathcal{T}\{x_1[n]\} + \beta \mathcal{T}\{x_2[n]\} \\
&= \alpha x_1^2[n] + \beta x_2^2[n] \neq (1) \quad \ldots (2)
\end{aligned}$$

$\therefore$ Not linear

Causality and stability

Causality:

i really had a good .... in bed yesterday

- Causal system: $y[n]$ depends only on past and present inputs, i.e. $x[k]$ for $k \leq n$
- Non-causal system: $y[n]$ depends on $x[k]$ with $k > n$

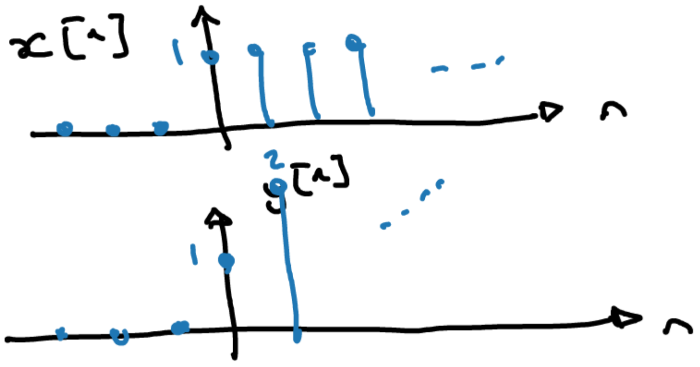
$y[n] = y[n-1] + x[n]$ → Causal

$y[n] = x[2n]$ → Non-causal

Bounded-input bounded-output (BIBO) stability:

if	$\lvert x[n] \rvert \leq M_x$ for all n
then	$\lvert y[n] \rvert \leq M_y$ for all n

Is the accumulator $y[n] = \sum_{k=-\infty}^{n} x[k]$ BIBO stable?



$y[n] \to \infty$