Energy and power of discrete signals

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Energy and power signals

Continuous signals:

$$E = \int_{\infty}^{\infty} v(t) i(t) \, \mathrm{d}t$$

$$= \int_{\infty}^{\infty} v^2(t) \, \mathrm{d}t$$

$1\,\Omega$

$$P = \lim_{t \to \infty} \frac{1}{2T} \int_{-T}^{T} v^2(t) \, \mathrm{d}t$$

Discrete signals:

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

if periodic with N:

$$P = \frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2$$

- if E finite => P = 0 discrete energy signal
- if P finite => Discrete Power signal

Parseval's theorem for discrete energy signals

Signal $x_1[n]$ has spectrum $\overset{\text{DTFT}}{X_1(\omega)}$ and $x_2[n]$ has spectrum $\overset{\text{DTFT}}{X_2(\omega)}$:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\omega)\cdot X_2^*(\omega)\,d\omega$$

DTFT

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\sum_{n=-\infty}^{\infty} x_1[n]\cdot e^{-j\omega n}\right]\cdot X_2^*(\omega)\cdot d\omega$$

$$= \sum_{n=-\infty}^{\infty} x_1[n]\,\frac{1}{2\pi}\int_{-\pi}^{\pi} X_2^*(\omega)\cdot e^{-j\omega n}\cdot d\omega$$

$$= \sum_{n=-\infty}^{\infty} x_1[n]\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \underline{X_2^*(-\omega_1)}\, e^{j\omega_1 n}\,d\omega_1\right]$$

$$= \sum_{n=-\infty}^{\infty} x_1[n]\, x_2^*[n]$$

$\omega_1 = -\omega$

$d\omega_1 = -d\omega$

ω	ω_1
π	$-\pi$
$-\pi$	π

Set $x_1[n] = x_2[n] = x[n]$

$$\sum_{n=-\infty}^{\infty} x[n]\cdot x^*[n]$$

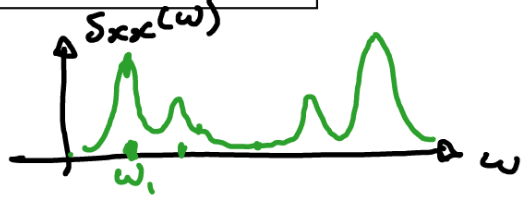
$$= \sum_{n=-\infty}^{\infty} |x[n]|^2 = E_x$$

$$\therefore\ E_x = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\cdot X^*(\omega)\cdot d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2\cdot d\omega$$

$$\boxed{E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2\, \mathrm{d}\omega}$$

Power density spectrum: $\boxed{S_{xx}(\omega) = |X(\omega)|^2}$



Parseval's theorem for discrete power signals

Periodic $x[n]$ with period N:

$$P_x = \frac{1}{N}\sum_{n=0}^{N-1}|x[n]|^2 = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\underbrace{x^*[n]}_{\text{IDFT}}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x[n]\left[\frac{1}{N}\sum_{k=0}^{N-1} X^*[N-k]e^{-j\omega k/N}\right]$$

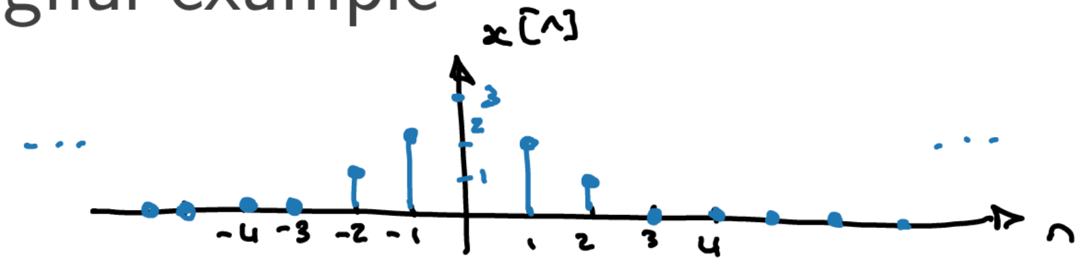
$$= \frac{1}{N}\sum_{n=0}^{N-1} X^*[n]\left[\underbrace{\frac{1}{N}\sum_{k=0}^{N-1} x[k]e^{-j\omega k/N}}_{\text{DFT}}\right] = \frac{1}{N^2}\sum_{n=0}^{N-1} X^*[n]X[n]$$

$$\boxed{P_x = \frac{1}{N}\sum_{n=0}^{N-1}|x[n]|^2 = \frac{1}{N^2}\sum_{k=0}^{N-1}|X[k]|^2}$$

Power density spectrum: $\boxed{S_{xx}[k] = \frac{1}{N}|X[k]|^2}$

Energy signal example

$$x[n] = \begin{cases} 3 - |n| & \text{if } |n| < 4 \\ 0 & \text{otherwise} \end{cases}$$



What are E and P?

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2 = 0^2 + 0^2 + \cdots + 0^2 + 1^2 + 2^2 + 3^2 + 2^2 + 1^2 + 0^2 + 0^2 + \cdots + 0^2$$

$$= 1 + 4 + 9 + 4 + 1$$

$$= 19$$

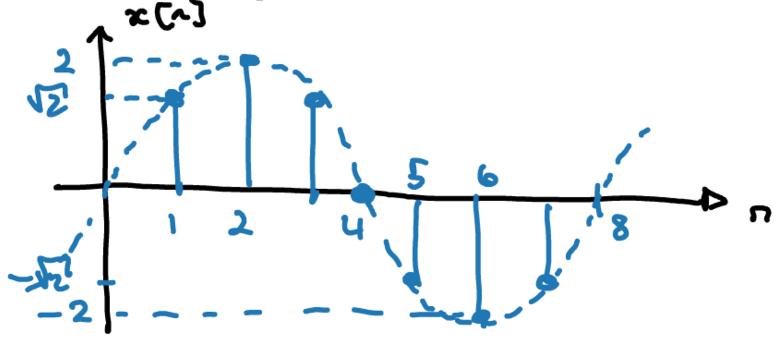
$$P = 0$$

Power signal example

$f_{\omega_0} = \frac{1}{8}$

$N = 8$

$$x[n] = 2\sin\left(\frac{\pi n}{4}\right) = 2\sin\left(2\pi \cdot \frac{1}{8} n\right)$$



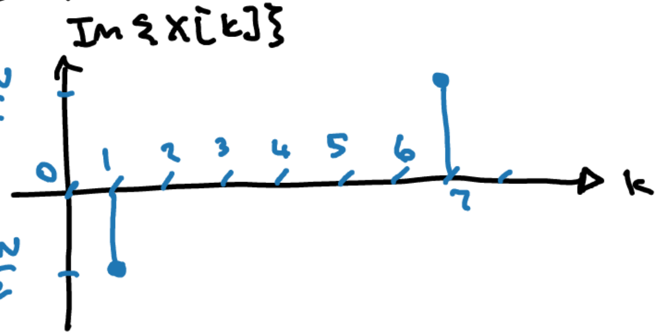
(a) What is E? $E = \infty$

(b) Calculate P in two ways.

① $P = \frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{8}\left(0+2+4+2+0+2+4+2+0\right) = \frac{16}{8} = 2$

② $P = \frac{1}{N^2}\sum_{k=0}^{N-1} |X[k]|^2$

Take 8-point DFT:



$$= \frac{1}{8^2}\left((-8)^2 + 8^2\right)$$

$$= \frac{2\cdot 8^2}{8^2} = 2$$

$$\frac{AN}{2} = \frac{2\cdot 8}{2} = 8$$