Correlation for discrete energy and power signals

Definitions, calculations, and applications

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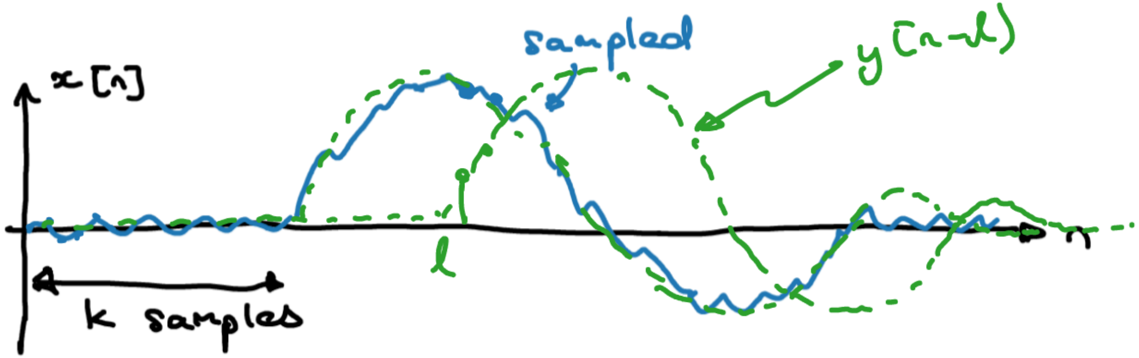
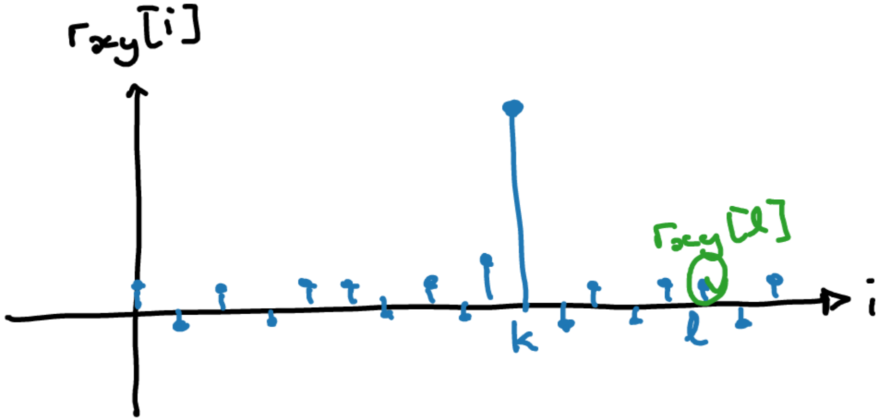
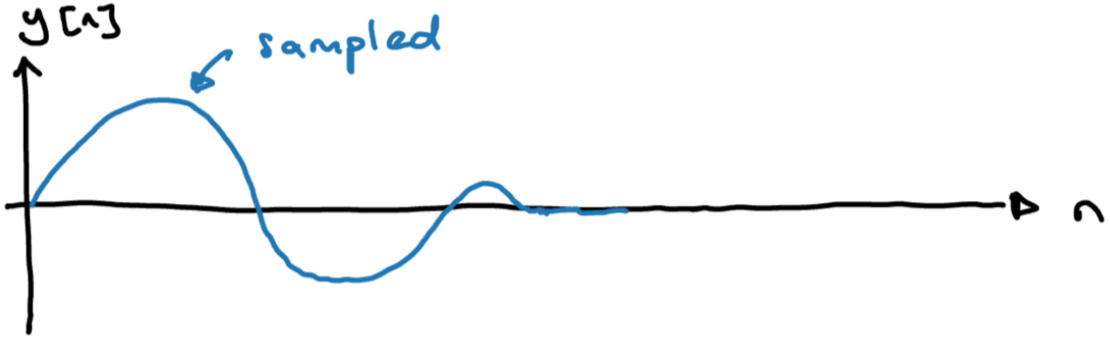
Cross-correlation of discrete energy signals

$$r_{xy}[i] = \sum_{n=-\infty}^{\infty} x[n]y[n-i] = \sum_{n=-\infty}^{\infty} x[n+i]y[n]$$







Properties of cross-correlation of energy signals

Cross-correlation is like convolution, but without reflection:

$$x[i] * y[i] = \sum_{n=-\infty}^{\infty} x[n]y[i-n]$$

$$\Rightarrow x[i] * y[-i] = \sum_{n=-\infty}^{\infty} x[n]y[n-i] = r_{xy}[i]$$

Cross-correlation in frequency domain:

$r_{xy}[i] = x[i] * y[-i]$

$$\mathcal{F}\left\{r_{xy}[i]\right\} = X(\omega)Y(-\omega)$$

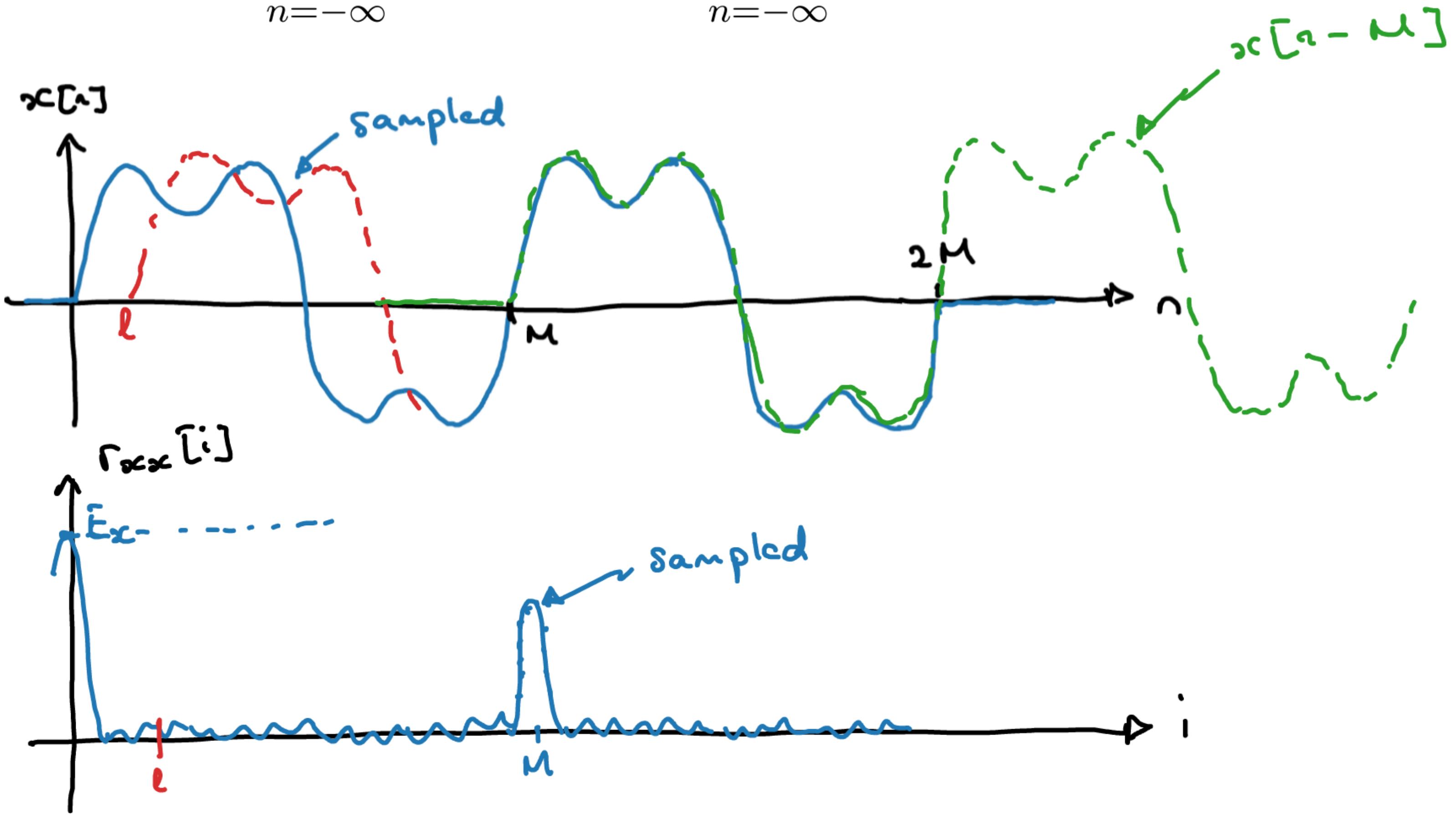
DTFT

Cross-correlation symmetry:

$$r_{yx}[i] = \sum_{n=-\infty}^{\infty} y[n+i]x[n] = \sum_{n=-\infty}^{\infty} x[n]y[n-(-i)] = r_{xy}[-i]$$

Autocorrelation of discrete energy signals

$$r_{xx}[i] = \sum_{n=-\infty}^{\infty} x[n]x[n-i] = \sum_{n=-\infty}^{\infty} x[n+i]x[n]$$



Properties of autocorrelation of energy signals

Autocorrelation symmetry:

$$r_{xx}[-i] = r_{xx}[i]$$

Autocorrelation and energy:

$$= \sum_{n=-\infty}^{\infty} x[n] \cdot x[n-0]$$

$$r_{xx}[0] = \sum_{n=-\infty}^{\infty} x^2[n] = E_x$$

Autocorrelation in frequency domain:

$$\mathcal{F}\left\{r_{xx}[i]\right\} = X(\omega)X(-\omega) = X(\omega)X^*(\omega)$$

$$\Rightarrow \mathcal{F}\left\{r_{xx}[i]\right\} = \left|X(\omega)\right|^2$$

Correlation of energy signals using DFT

Recall that $r_{xy}[i] = x[i] * y[-i]$ when $x[n]$ and $y[n]$ are energy signals

$N \geqslant L+P-1$

Zero pad $x[n]$ and $y[n]$ appropriately

Cross-correlation via DFT:

$$\tilde{X}[k] = \mathrm{DFT}\left\{\tilde{x}[i]\right\}$$

$$Y[k] = \mathrm{DFT}\left\{y[i]\right\} \Rightarrow \mathrm{DFT}\left\{y[-i]\right\} = Y^*[k]$$

$$\mathrm{DFT}\left\{r_{xy}[i]\right\} = X[k]\,Y^*[k]$$

Bounds

Can proove that:

$$|r_{xx}[i]| \leq r_{xx}[0] = E_x$$

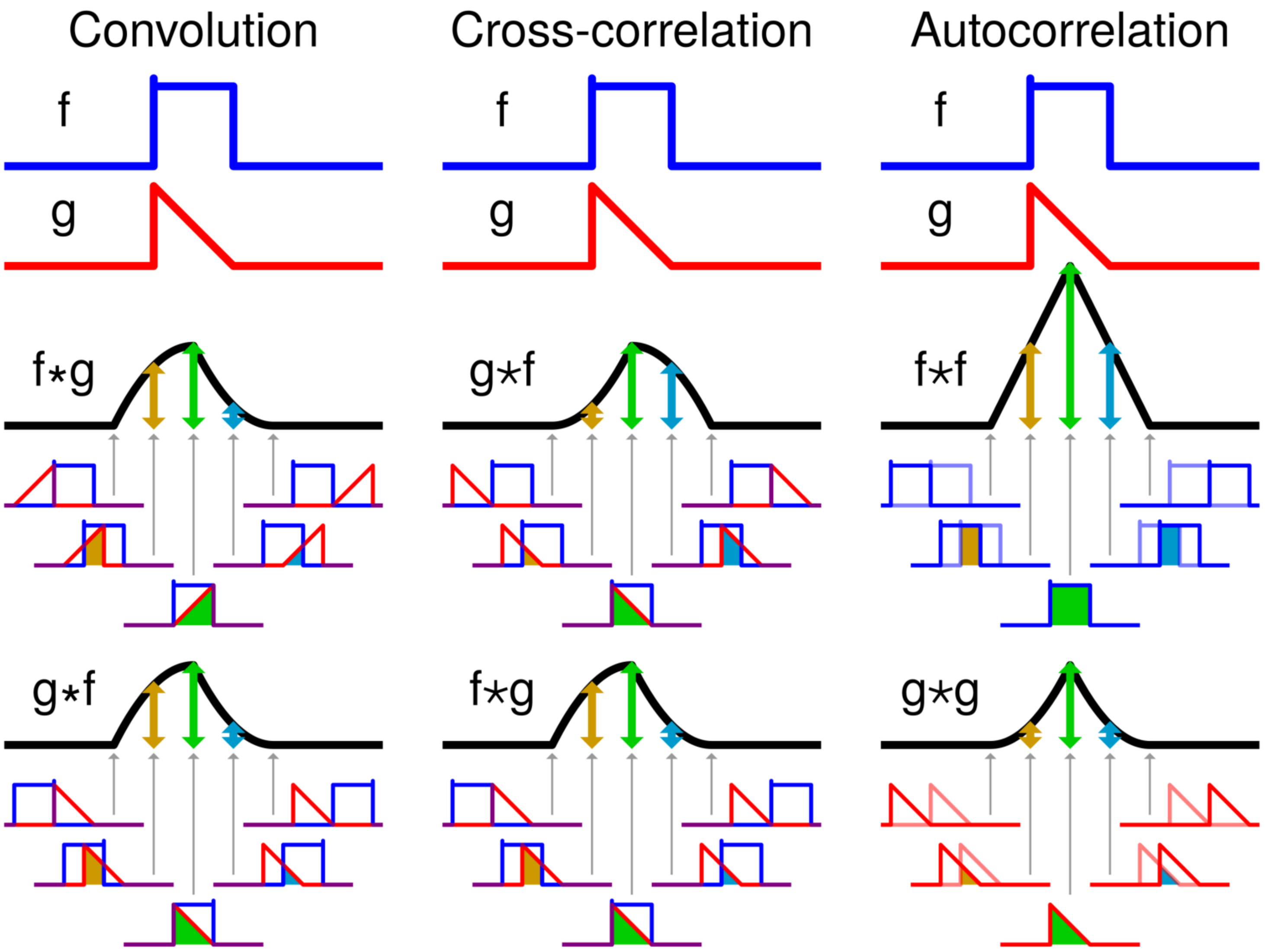
and similarly that:

$$|r_{xy}[i]| \leq \sqrt{r_{xx}[0] r_{yy}[0]} = \sqrt{E_x E_y}$$

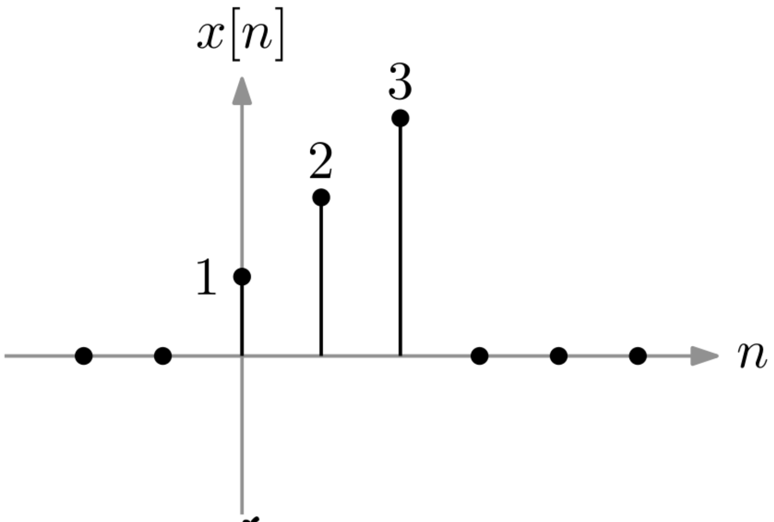
Often scale by upper bounds:

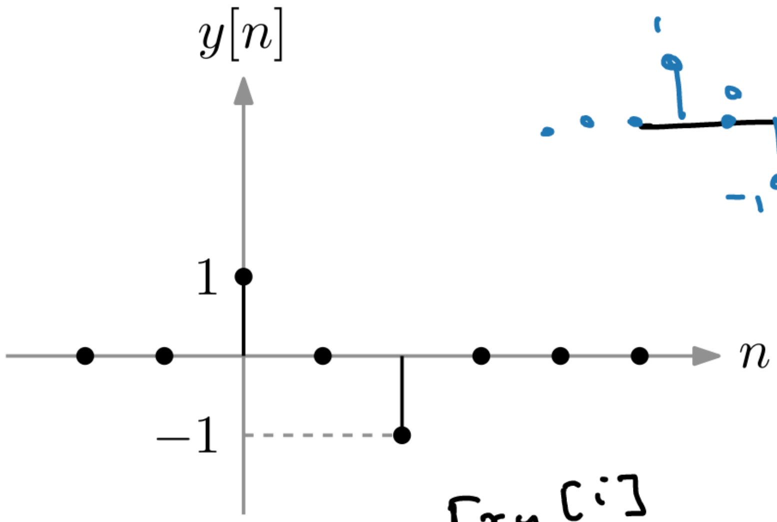
$$\rho_{xx}[i] = \frac{r_{xx}[i]}{E_x} \qquad -1 \leq \rho_{xx}[i] \leq 1$$

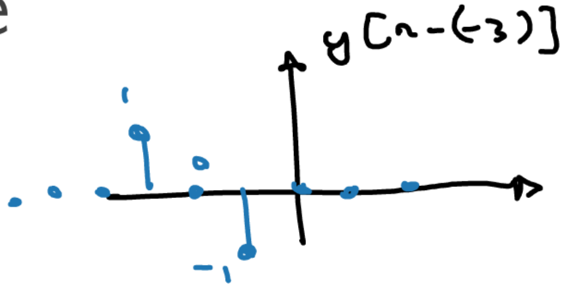
$$\rho_{xy}[i] = \frac{r_{xy}[i]}{\sqrt{E_x E_y}} \qquad -1 \leq \rho_{xy}[i] \leq 1$$


Convolution
Cross-correlation
Autocorrelation
f
g
f∗g
g∗f
g∗f
f∗g
f∗f
g∗g

Cross-correlation example







$$r_{xy}[i] = \sum_{n=-\infty}^{\infty} x[n] \cdot y[n-i]$$

$$r_{xy}[-3] = 0$$

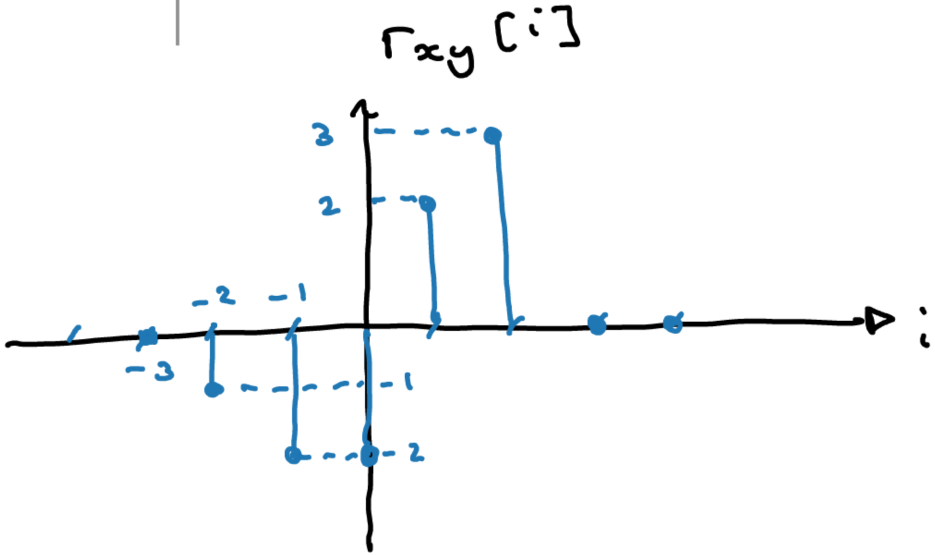
$$r_{xy}[-2] = -1$$

$$r_{xy}[-1] = -2$$

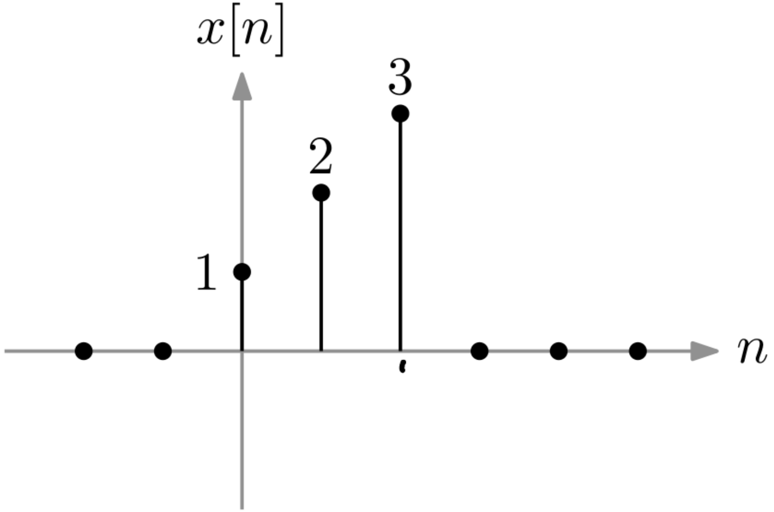
$$r_{xy}[0] = 1 - 3 = -2$$

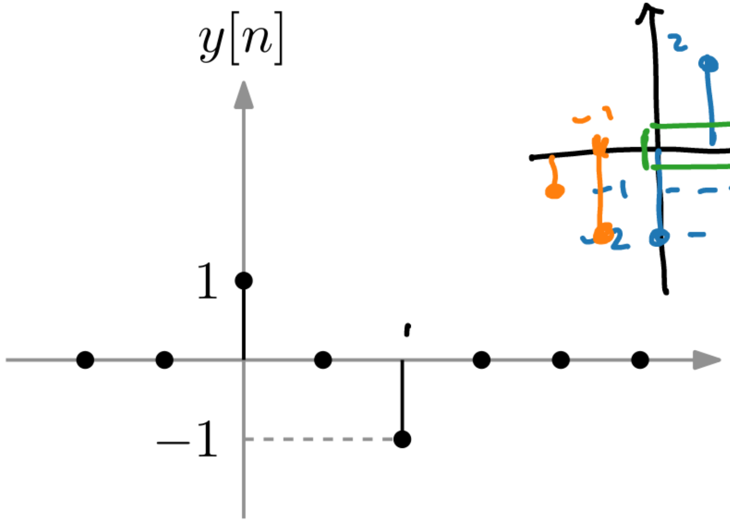
$$r_{xy}[1] = 2$$

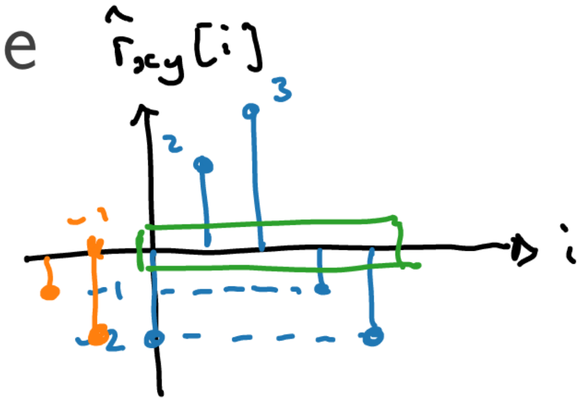
$$r_{xy}[2] = 3$$



Cross-correlation example







$$N \geq L + P - 1 = 3 + 3 - 1 = 5$$

$$x[n] = \{\underset{\uparrow}{1} \quad 2 \quad 3\}$$

$$x_{zp}[n] = \{\underset{\uparrow}{1} \quad 2 \quad 3 \quad 0 \quad 0\}$$

$$y_{zp}[n] = \{\underset{\uparrow}{1} \quad 0 \quad -1 \quad 0 \quad 0\}$$

$$X = \text{np.fft.fft}(x_{zp})$$

$$Y = \text{np.fft.fft}(y_{zp})$$

$$\hat{r}_{xy} = \text{np.fft.ifft}(X \cdot \text{np.conj}(Y))$$

$\boxed{\text{Demo}}$

x_1 : Blaues Licht

x_2 : Lovejoy

x_3 :

⋮

y : Sample

r_{x_1y}

r_{x_2y}

Correlation of power signals

Cross-correlation of power signals:

$$r_{xy}[i] = \lim_{M \to \infty} \frac{1}{2M+1} \sum_{n=-M}^{M} x[n]y[n-i]$$

Autocorrelation of power signals:

$$r_{xx}[i] = \lim_{M \to \infty} \frac{1}{2M+1} \sum_{n=-M}^{M} x[n]x[n-i]$$

Correlation of periodic signals with period N:

$$r_{xy}[i] = \frac{1}{N} \sum_{n=0}^{N-1} x[n]y[n-i]$$

$$r_{xx}[i] = \frac{1}{N} \sum_{n=0}^{N-1} x[n]x[n-i]$$

Autocorrelation and power: $r_{xx}[0] = P_x$

Always

Cross-correlation is like circular convolution, but without reflection:

$$x[i] \underset{N}{\circledast} y[i] = \sum_{n=0}^{N-1} x[n]\tilde{y}[i-n]$$

$$\Rightarrow x[i] \underset{N}{\circledast} y[-i] = \sum_{n=0}^{N-1} x[n]\tilde{y}[\overset{n-i}{\cancel{i-n}}]$$

$$= N r_{xy}[i]$$

Cross-correlation in frequency domain:

$$N \cdot \mathrm{DFT}\left\{r_{xy}[i]\right\} = \mathrm{DFT}\left\{x[i]\right\} \mathrm{DFT}\left\{y[-i]\right\} = X[k]\, Y^*[k]$$

Autocorrelation in frequency domain:

$$N \cdot \mathrm{DFT}\left\{r_{xx}[i]\right\} = X[k]\, X^*[k] = |X[k]|^2 = N S_{xx}[k]$$

x[n] and y[n] that are periodic with period N

Bounds:

$$|r_{xx}[i]| \leq r_{xx}[0] = P_x$$

$$|r_{xy}[i]| \leq \sqrt{r_{xx}[0] r_{yy}[0]} = \sqrt{P_x P_y}$$

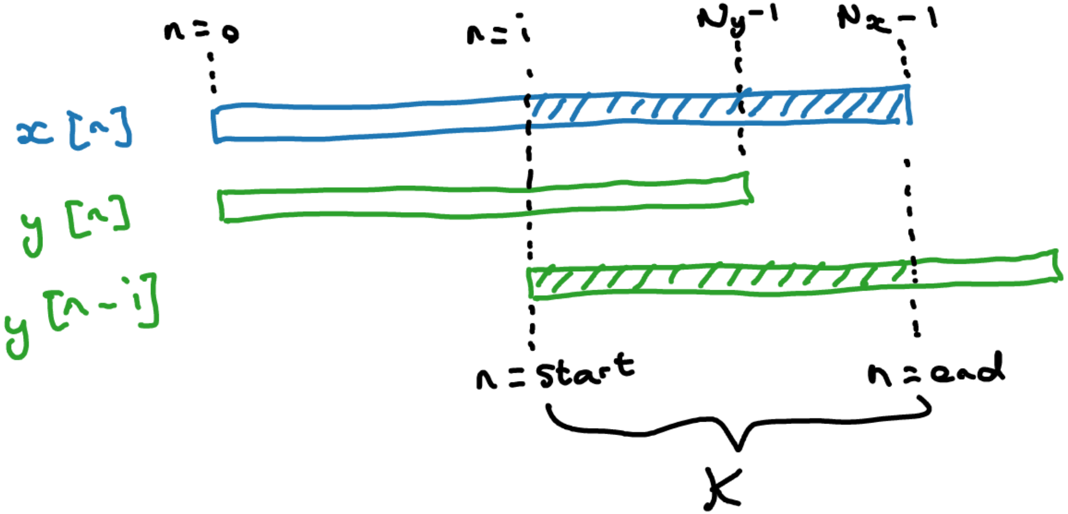
Estimating cross-correlation of power signals from windows

Cross-correlation of power signals:

$$r_{xy}[i] = \lim_{M\to\infty} \frac{1}{2M+1} \sum_{n=-M}^{M} x[n]y[n-i]$$

Normally we only have a finite window of each signal:

$$r_{xy}[i] \approx \frac{1}{|\mathcal{X}|} \sum_{n \in \mathcal{X}} x[n] \cdot y[n-i]$$



$= \{\text{start}, \text{start}+1, \dots, \text{end}\}$

```
for i = some number:
    for n in range(start, end):
        r_xy[i] += x[n] · y[n-i]
```

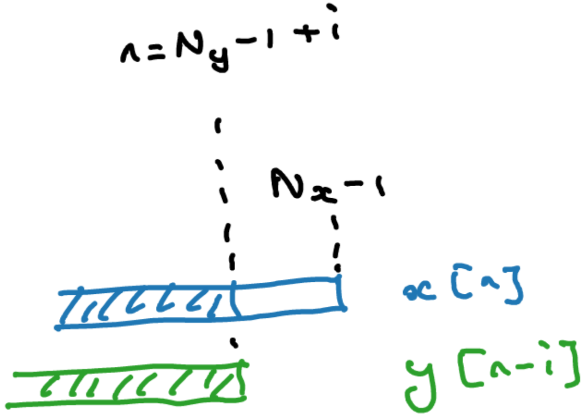
Where does $\mathcal{K}$ start?

- When $i < 0$? start $= 0$
- When $i > 0$? start $= i$

$\max(0, i)$

Where does $\mathcal{K}$ end?

- When $y[n-i]$ ends after $x[n]$? end $= N_x - 1$
- When $y[n-i]$ ends before $x[n]$? end $= N_y - 1 + i$



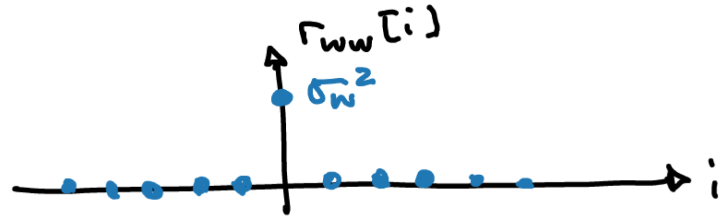
$\min(N_x - 1, N_y - 1 + i)$

Therefore: $\mathcal{K} = \{\max(0, i), \ldots, \min(N_x - 1, N_y - 1 + i)\}$

Detecting periodicity using correlation

period N ?

$$y[n] = x[n] + w[n]$$



$$r_{yy}[i] = \lim_{M\to\infty} \frac{1}{2M+1} \sum_{n=-M}^{M} y[n]\cdot y[n-i]$$

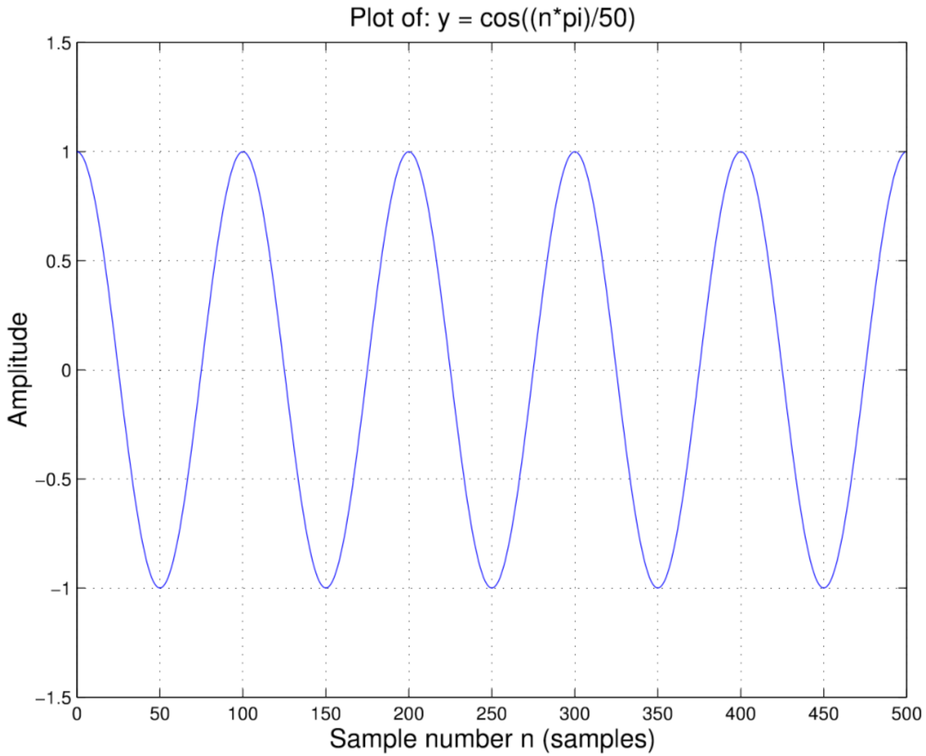
$$= \lim_{M\to\infty} \frac{1}{2M+1} \sum_{n=-M}^{M} \left(x[n] + w[n]\right)\cdot\left(x[n-i] + w[n-i]\right)$$

$$= \lim_{M\to\infty} \frac{1}{2M+1} \left\{ \sum_{n=-M}^{M} x[n]\cdot x[n-i] + \sum_{n=-M}^{M} x[n]\cdot w[n-i] + \sum_{n=-M}^{M} w[n]\cdot x[n-i] + \sum_{n=-M}^{M} w[n]\cdot w[n-i] \right\}$$

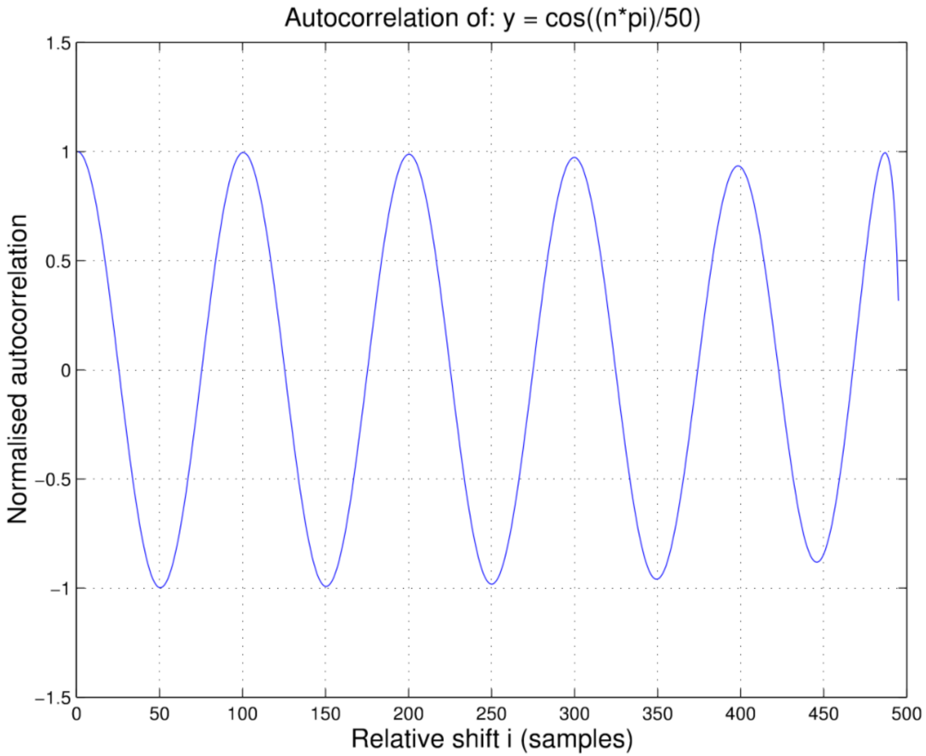
$$= r_{xx}[i] + \underbrace{r_{xw}[i]}_{\approx 0} + \underbrace{r_{wx}[i]}_{\approx 0} + r_{ww}[i]$$

When $i \neq 0$: $\quad r_{yy}[i] = r_{xx}[i]$

To find period: Look for peaks in $r_{yy}[i]$



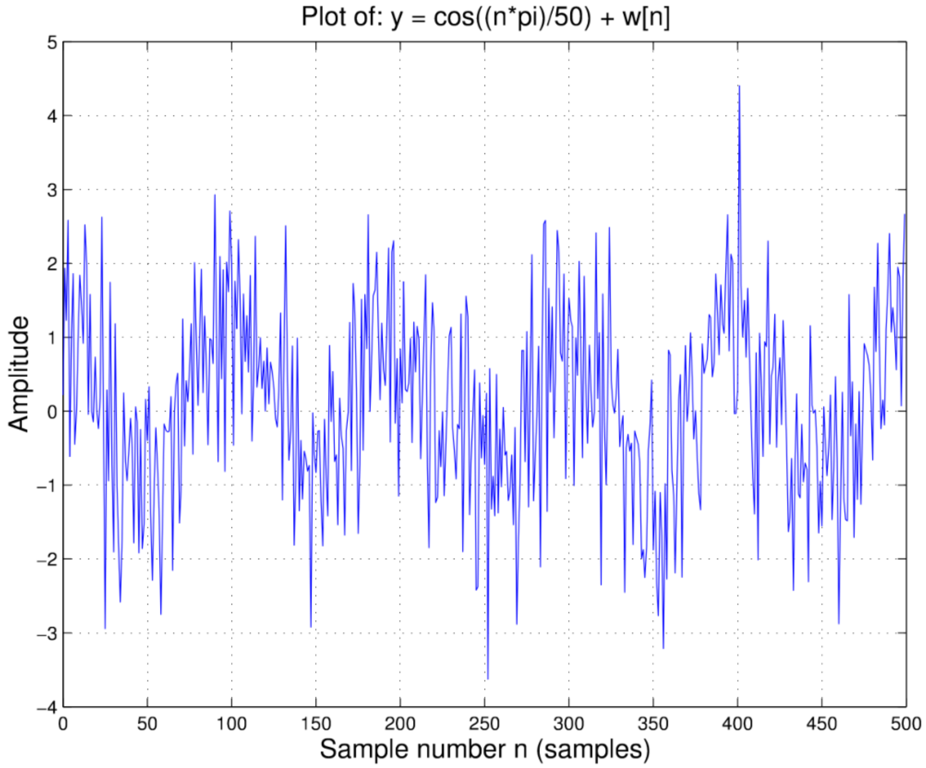
Plot of: y = cos((n*pi)/50)
Amplitude
Sample number n (samples)
1.5
1
0.5
0
−0.5
−1
−1.5
0
50
100
150
200
250
300
350
400
450
500

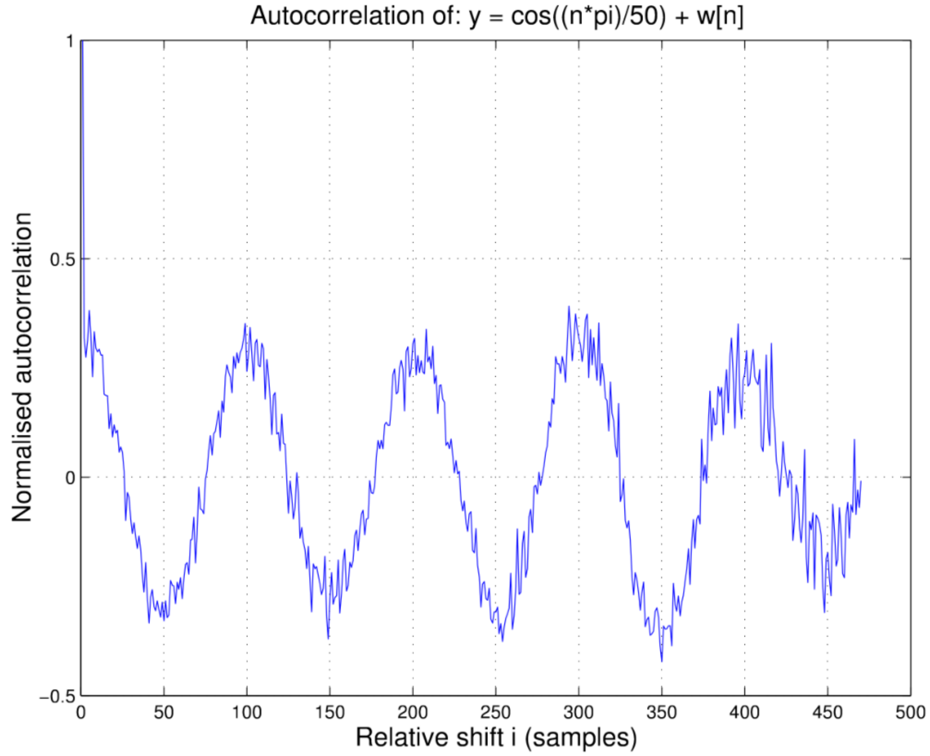


Autocorrelation of: y = cos((n*pi)/50)
Normalised autocorrelation
Relative shift i (samples)
1.5
1
0.5
0
−0.5
−1
−1.5
0
50
100
150
200
250
300
350
400
450
500

$y[n] = x[n] + w[n]$

$r_{yy}[i]$





$2M+1 \gg N$